An Introduction to

Intuitionistic Fuzzy Spatial Region

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Abstract. In Geoinformation systems (GIS) there is a need to model spatial regions with indeterminate boundary and under uncertainties. Although fuzzy logic methods are of great interest in many GIS applications, the traditional fuzzy logic has two important deficiencies. First, to apply the fuzzy logic, we need to assign, to every property and for every value, a crisp membership function. Second, fuzzy logic does not distinguish between the situation in which there is no knowledge about a certain statement and a situation that the belief to the statement in favor and against is the same. In order to solve these problems, we motivate to use intuitionistic fuzzy logic. This paper gives fundamental concepts and properties of an intuitionistic fuzzy spatial region. We introduce a new theoretical framework and discuss possible contributions to different fields. We will link our suggested method to other models like the rough set theory.

1 Introduction

The development of general theories of uncertainty modeling of spatial objects and spatial relations has recently found increased attention in GIS. Describing spatial phenomena suffers from uncertainty. Sources of uncertainty are the inexact or incomplete definition of objects, and the inability to observe precise and complete relevant data (see (Burrough and Frank 1996)). The description of objects-static or dynamic- is not only uncertain in the above mentioned sense, but also contradictory in different contexts(Kokla and Kavouras 2001).

Although fuzzy logic methods are of great interest in many GIS applications, the traditional fuzzy logic has two important deficiencies. First, to apply the fuzzy logic, we need to assign, to every property and for every value, a crisp membership function. Second, fuzzy logic does not distinguish between the situation in which there is no knowledge about a certain statement and a situation that the belief to the statement in favor and against is the same. Due to this fact, it is not recommended for problems with missing data and where grades of membership are hard to define (Roy 1999). Using not enough or irrelevant data like aged satellite images are one example for the aforementioned problem. Another example is the definition of objects. They can be very different for the same object. Some experts may have different views of an object e.g., 'forest'. This problem is emerging whenever one has to deal with interoperability of different systems, combining different data sets.

The paper will discuss several possible contributions to the GIS field including remote sensing, object reconstruction from airborne laser scanner, real time tracking, routing applications and modeling cognitive agents.

One of most important characteristics of qualitative properties of spatial data and perhaps the most fundamental aspect of space is topology and topological relationship. Topological relations between spatial objects like meet and overlap are such relationships that are invariant with respect to specific transformations due to homeomorphism.

In this paper, a simple fuzzy region and fundamental concepts for uncertainty modeling of spatial relationships are analyzed from the view point of intuitionistic fuzzy (IF) logic. We demonstrate how it can provide a model for fuzzy region; i.e., regions with indeterminate boundaries. The remainder of this paper is structured as follows: Section 2 reviews the related work. Section 3 introduces necessary concepts. The contributions of intuitionistic fuzzy logic to some problems in GIS are discussed in this section. Section 4 proposes novel concepts and their properties to define a simple intuitionistic fuzzy spatial region (IFSR). The concluding section 5 gives a summary and an outlook to future research topics.

2 Related Work

Algebraic topological models for spatial objects was introduced in (White 1979). Thirteen topological relations between two temporal intervals were identified by (Allen 1983).

After the 4-intersection model (Egenhofer 1989; Egenhofer and Franzosa 1991) the 9-intersection approach (Egenhofer and Herring 1991) was proposed as a formalism for topological relations. This approach is based on point-set topological concepts. In the 9-intersection method, a spatial object A is decomposed into three parts: an interior denoted by A^o, an exterior denoted by A^E, and a boundary denoted by ∂A. There are nine intersections between six parts of two objects. The other significant approach known as RCC (Region-Connection Calculus) has been provided by Cohn et al. (Randell, Cui et al. 1992; Gotts, Gooday et al. 1995; Cohn, Bennet et al. 1997).

During recent years, the topological relations have been extended into fuzzy domains. An example of a fuzzy object was provided by (Fisher 1996). A number of papers (Schneider 1999; Schneider 2000; Schneider 2001; Schneider 2001) was presented to model fuzzy set in GIS community and to design a system of fuzzy spatial data types including operations and predicates. Molenaar (Molenaar 1998) extended the formal model into fuzzy domain and based on this model Cheng (Cheng 1999) proposed a process-oriented spatio-temporal data model. The intersection model is extended to vague regions by three main approaches: the work of Clementini and Di Felice (Clementini and Di Felice 1996; Clementini and Di Felice 1997) on regions with "broad boundary", the work of Zhan (Zhan 1998) who developed a method for approximately analyzing binary topological relations between geographic regions with indeterminate boundaries based on fuzzy sets, and Tang and Kainz (Tang and Kainz 2002) that provided a 3*3, a 4*4, and a 5*5 intersection matrix based on different topological parts of two fuzzy regions. The extension of the RCC schemes to accommodate vague region has been addressed by Lehmann and Cohn (Lehmann and Cohn 1994), and by Cohn and Gotts (Cohn and Gotts 1996). In this direction Stell and Worboys (Stell and Worboys 1997) have used Heyting structures.

The notion of intuitionistic fuzzy sets (IFS) was introduced by Atanassov (Atanassov 1986; Atanassov 1989; Atanassov 1999) as a generalization of fuzzy sets. Later the concept of intuitionistic fuzzy topology was introduced by Coker (Coker 1997).

3 The Contribution of IFS

3.1 Preliminaries

First we present the fundamental concepts and definitions given by Atanassov.

Definition 3.1 (Atanassov (Atanassov 1999)): Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS) A in X is an object having the following form

$$A := \{ < x, \mu_A(x), \nu_A(x) > | x \in X \}$$

where the function $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$, respectively. For every $x \in X$, μ_A and ν_A satisfy:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

In contrary of traditional fuzzy, the addition of μ_A and ν_A does not necessarily have to be 1. This is particularly useful when system may lack complete information.

Definition 3.2 (Atanassov (Atanassov 1999)): For every two IFSs A and B, the following operations can be defined:

$$A \subseteq B \text{ iff } (\forall x \in X)(\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x));$$
$$A = B \text{ iff } (\forall x \in X)(\mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x));$$
$$A^c := \quad \{< x, \nu_A(x), \mu_A(x) > | \, x \in X\};$$
$$A \cap B := \quad \{< x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) > | \, x \in X\};$$
$$A \cup B := \quad \{< x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) > | \, x \in X\};$$
$$\bigcup A_i := \quad \{< x, \max(\mu_i), \min(\nu_i) > | \, x \in X\} \quad i \in I;$$
$$\bigcap A_i := \quad \{< x, \min(\mu_i), \max(\nu_i) > | \, x \in X\} \quad i \in I;$$
$$[\,]A := \quad \{< x, \mu_A(x), 1 - \mu_A(x) > | \, x \in X\} \quad \text{(Necessity operator)};$$
$$\langle\rangle A := \quad \{< x, 1 - \nu_A(x), \nu_A(x) > | \, x \in X\} \quad \text{(Possibility operator)}.$$

3.2 Possible Applications

Given the definitions of the previous section several possible contributions are discussed. Intuitionistic fuzzy logic may be used to solve some of the problems addressed.

Remote sensing: In satellite images some pixels might remain unclassified due to clouds or absorption of the sensors signal. Missing data causes problems in the classification of pixels. Figure 1 shows an example of an intuitionistic fuzzy region in a raster environment, where the degree of membership and the degree of non-membership are the value of each pixel. Note the difference in the resulting object boundaries.

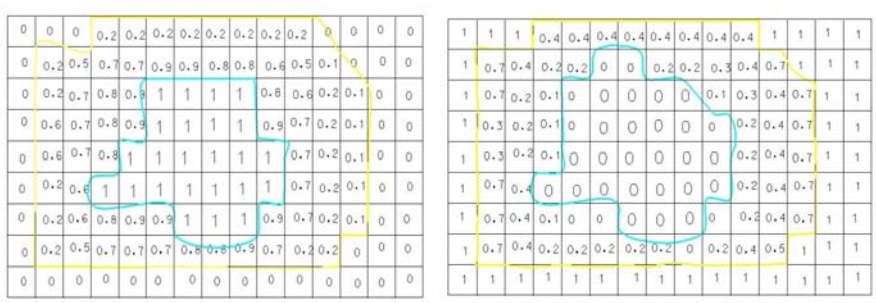


Fig. 1. An IFSR in raster environment. In the left image memberships and in the right image non-memberships are listed. It can be seen that the addition of these two values for a certain pixel need not to be one.

One major source of uncertainty in satellite images is age. For planning purposes images covering the area of BAM in Iran have been made. After the recent earthquake there is a need for a new classification. The procedure will face cases in which arguments indicate that part of e.g. a grassland area did not change and some arguments against. Frequently there occurs the need to combine aged images with new data.

Object reconstruction using Laser scanning: The success of classification tasks is often impeded by uncertainty. When scanning the surface of the landscape with an airborne laser scanner, point clouds can be obtained. For the further processing points are often classified as belonging to natural objects, manmade objects or the ground surface. To model objects like buildings a two step process is usually applied. First, points are detected that belong to building regions. Second, the identified point clusters are used to reconstruct a model of a building. One of the main impediments for this method is vegetation. Fig. 2 shows a case were points belong the same time to a building as to vegetation. Current methods will ignore these points from further processing. The intuitionistic approach allows to handle this points extra. Their belonging to the building as well as not belonging can be modeled by applying weights for both states. A consideration in the further reconstruction process is possible.

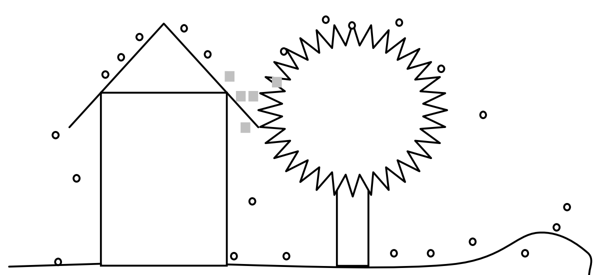
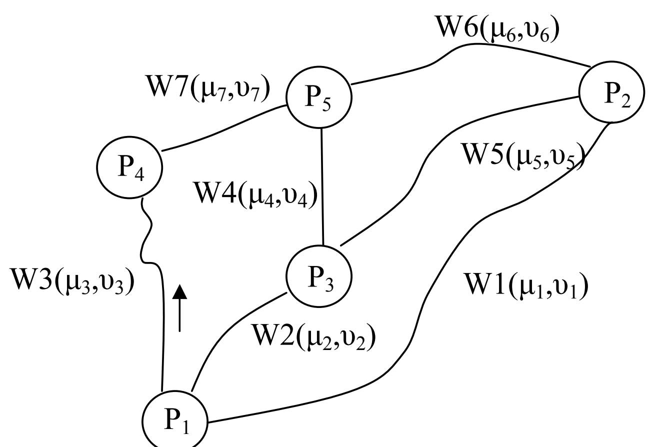


Fig. 2. The grey boxes mark points that have both arguments to be points of vegetation as well as manmade object. The points could be attributed as belonging to the house at the same time as not belonging.

Routing: In the GEORAMA project carried out under the funding of the EC a portal with web services for tourists in mountain and countryside area has been investigated. Tourists are very often interested in routing applications like city guides or trip planners in mountainous area. Existing services can not handle very well the preferences a single user has. Therefore these services are personalized. The user has to provide some input about his preferences, e.g., what he likes to eat, how much money he likes to spend, his physical condition etc. These values are in most cases not crisp.

Usually graphs are used to model networks of routes. A path may be modeled by a weighted graph, where weights might be the distance between two nodes. In practice it is not always possible neither to assign a certain weight to an arc nor to a node. Mobile applications could benefit by the definition of intuitionistic fuzzy graphs and networks. Each node or arc might have a membership degree to be interesting for a hiker and a non membership to be not interesting. A node is interesting because there is a restaurant, a famous sculpture or a natural beauty. A node might be at the same time non interesting because it is very crowded, or the available restaurant is very expensive. Defining weights by two membership functions may help to determine the path in a network that fits the most to the preferences of a tourist (see Fig. 3).

Note that it is necessary that weights some up to 1 in the intuitionistic approach as seen in Fig. 3 for the Weight W_2 of the connection between P_1 and P_3.



From	To	Wi	μ_i	υ_i
P_1	P_2	1	0.7	0.3
P_1	P_3	2	0.1	0.5
P_1	P_4	3	0.9	0.0
…	…	…	…	…

Fig. 3. Definition of an IF network with IF weights for routing applications.

Mental models: The IF logic seems a step closer to a model of the human thinking process. Humans are in doubt, in certain situations e.g., when they search for an object. When an object disappeared, different strategies can be applied to find the object, as already seen in the observation of children (Gopnik and Meltzoff 1997). The object may be occluded; it may be on the place where it was seen last time visible or at the place first time visible. The object may also have moved on a trajectory. All these hypothesis and alternatives might be weighted as being plausible and non plausible. Introducing membership functions for being plausible as well as non plausible at the same time may improve the model of a cognitive agent. The intuitionistic view point may help to model doubts in the decision process.

Real time tracking: Another example is the real-time tracking of a moving agent. Usually, the ideal abstract of computing the function f, is the equation y=f(x), where x stands for the input and y for the output. In practice, there are some delays between blocks of procedures. For example in navigation the system works with a little while ago position, not really at the same time. Introducing detailed formula considering time needs more complexity and extra time operators. From an engineering point of view, however, such a detailed low-level analysis, though it can be done, is not necessarily a good idea. The intuitionistic viewpoint can be applied in this scenario.

Membership function: An important problem in fuzzy logic is finding the right membership function. Expert knowledge or empirical investigations are necessary to trigger the functions. A decrease in efforts for defining the membership functions can be expected using the intuitionistic approach. The knowledge provided can be coarser, as uncertainties can be expressed in two ways.

3.3 Linking IF with other Models

In this section the relation of IF model with other models will be shown. It can be easily seen that some different models derived from this model. So the intuitionistic fuzzy logic is a very general approach and links to other well known models.

First order fuzzy set: As explained before, if $\nu = 1-\mu$ then the traditional fuzzy logic is derived.

Interval-valued fuzzy set: In this fuzzy method the membership degree of x in A is given as the interval $[\mu_1, \mu_2]$. So by defining $\mu = \mu_1$ and $\nu = 1-\mu_2$ a valid membership and non-membership degree for x in an IFS can be derived.

Rough set: Rough set theory is suited to use data with a partition or in cases where data may be missing (Roy 1999). Such sets can be described by a discrete form of IFS which is called IF special set (IFSS)(Coker 1996). Namely, an IFSS A is defined by $A=<X, A_1, A_2>$ where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. A_1 is the set of members of A and A_2 is the set of non members of A.

Four-valued logic: This kind of logic like Belnap's logic which proposed to deal with inconsistency can be derived by discarding the constraint $\mu + \nu \leq 1$.

4 Some Topological Notions of Fuzzy Region

Definition 4.1 (Coker (Coker 1997)):

We define an empty set $0_{\sim}$ and a non empty set $1_{\sim}$ as follows:

$0_{\sim} := \{<x,0,1> \mid x \in X\}$ $\qquad$ $1_{\sim} := \{<x,1,0> \mid x \in X\}$.

Consequently, an intuitionistic fuzzy topology (IFT) on a nonempty set X is a family T of IFSs in X satisfying the following axioms:

$(T1)\ 0_{\sim}, 1_{\sim} \in T;$

$(T2)\ G_i \cap G_j \in T\ \textit{for any}\ G_i, G_j \in T;$

$(T3) \bigcup G_i \in T\ \textit{for arbitrary family}\ \{G_i \mid i \in I\} \subset T.$

The pair (IFS(X),T) is called an intuitionistic fuzzy topological space (IFTS) and any IFS in T is known as an intuitionistic fuzzy open set (IFOS) in X. The complement A^c is called an intuitionistic fuzzy closed set (IFCS) in X. In a next step, a simple IF spatial region (IFSR) will be introduced.

Definition 4.2(Coker 1997) : Let (*X,T*) be an IF topological space and $A = <x, \mu_A, \nu_A>$ be an IFS in *X*. Then the fuzzy interior and fuzzy closure are defined by:

$$A^- = \bigcap\{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\},$$

$$A^o = \bigcup\{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}.$$

Corollary 4.1(Coker 1997):

$$A^o \cap B^o = (A \cap B)^o, A^- \cup B^- = (A \cup B)^-$$

$$A^o \subseteq A \subseteq A^-$$

$$A^{oc} = A^{c-}, A^{-c} = A^{co}.$$

Now, we add some further definitions and propositions.

Definition 4.3: We define an IF boundary (IFB) of an IFS $A = <x, \mu_A, \nu_A>$ by:

$$\partial A = A^- \cap A^{c-}.$$

The following theorem which is inspired by (Tang and Kainz 2002) shows the intersection methods no longer guarantees a unique solution.

Corollary 4.2: $\partial A \cap A^o = 0_\sim$ iff A^o is crisp (i.e., $A^o = 0_\sim \text{ or } A^o = 1_\sim$).

Proof. =>) If $A^o = \{< x, 0 < \mu_{A^o} < 1, 0 < \nu_{A^o} < 1 > | x \in X\}$, then $A^- = \{< x, 0 < \mu_{A^-} \le 1, 0 \le \nu_{A^-} < 1 > | x \in X\}$ and $A^{oc} = \{< x, 0 < \nu_{A^o} < 1, 0 < \mu_{A^o} < 1 > | x \in X\}$. Then, $\partial A \cap A^o = A^- \cap A^{c-} \cap A^o = A^- \cap A^{oc} \cap A^{c-} = \{x, 0 < \min(\mu_{A^o}, \mu_{A^-}, \nu_{A^-}) < 1,$
$0 < \max(\nu_{A^o}, \nu_{A^-}, \mu_{A^o}) < 1 | x \in X\}$.
Therefore, if $\partial A \cap A^o = 0_\sim$, then $A^o = 0_\sim$ or $A^o = 1_\sim$.

<=) If A^o is crisp, then $A^o = 0_\sim$ or $A^o = 1_\sim$. If $A^o = 1_\sim$ then $A^{oc} = 0_\sim$ and $A^- = 1_\sim$, then $\partial A \cap A^o = 0_\sim$. If $A^o = 0_\sim$ then it immediately results that $\partial A \cap A^o = 0_\sim$.

Definition 4.4: Let $A = < x, \mu_A, \nu_A >$ be an IFS in (*X,T*). Suppose that the family of IFOS's contained in A are indexed by the family $< x, \mu_{G_i}, \nu_{G_i} >: i \in I\}$ and the family of IFOS's containing A are indexed by the family $< x, \mu_{K_j}, \nu_{K_j} >: j \in J\}$. Then two interiors, closures, and boundaries are defined as following:

$A^o_{[]} := < x, \max(\mu_{G_i}), \min(1-\mu_{G_i} >$ $\quad A^o_{\langle\rangle} := < x, \max(1-\nu_{G_i}), \min(\nu_{G_i}) >$

$A^-_{[]} := < x, \min(\mu_{K_j}), \max(1-\mu_{K_j}) >$ $\quad A^-_{\langle\rangle} := < x, \min(1-\nu_{K_j}), \max(\nu_{K_j}) >$

$\partial A_{[]} := A^-_{[]} \cap A^{c-}_{[]}$ $\quad \partial A_{\langle\rangle} := A^-_{\langle\rangle} \cap A^{c-}_{\langle\rangle}$.

Proposition 4.1:

(a) $A^o_{[]} \subseteq A^o \subseteq A^o_{\langle\rangle}$ (b) $A^-_{[]} \subseteq A^- \subseteq A^-_{\langle\rangle}$

(c) $A^o_{\{[],\langle\rangle\}} = \{[],\langle\rangle\}A^o$ *and* $A^-_{\{[],\langle\rangle\}} = \{[],\langle\rangle\}A^-$

Proof. We shall only prove (c), and the others are obvious.

$[]A^o = < x, \max(\mu_{G_i}), 1-\max(\mu_{G_i}) >$. Based on knowing that $1-\max(\mu_{G_i}) = \min(1-\mu_{G_i})$, then $[]A^o = < x, \max(\mu_{G_i}), \min(1-\mu_{G_i}) > = A^o_{[]}$. In a similar way the others can be proved.

Proposition 4.2:

(a) $(A^o_{\{[],\langle\rangle\}})_{\{[],\langle\rangle\}} = A^o_{\{[],\langle\rangle\}}$ (b) $(A^-_{\{[],\langle\rangle\}})_{\{[],\langle\rangle\}} = A^-_{\{[],\langle\rangle\}}$

Proof. From proposition 4.1(c) we have $A^o_{\{[],\langle\rangle\}} = \{[],\langle\rangle\}A^o$ and knowing that $[]([]B) = []B$ (the same is valid for $\langle\rangle B$)(Atanassov 1999),then $(A^o_{\{[],\langle\rangle\}})_{\{[],\langle\rangle\}} = A^o_{\{[],\langle\rangle\}}$.

Definition 4.5: Let $A = < x, \mu_A, \nu_A >$ be an IFS in (*X,T*). We define exterior of *A* as follows:

$$A^E = X \cap A^c.$$

The introduced concepts can be further demonstrated with the figures below. A traditional fuzzy region and an IFSR are shown in Figure 4.

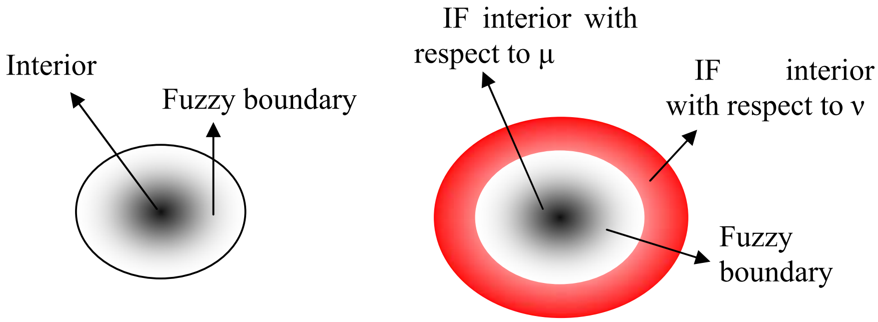


Fig . 4: A region in traditional and intuitionistic fuzzy viewpoints.

Definition 4.6:

An IFOS A is called *regular open* iff $A = A^{-o}$, and

An IFCS A is called *regular closed* iff $A = A^{o-}$.

Now, we shall obtain a formal model for a simple spatial fuzzy region based on IF connectedness defined in (Coker 1997).

Definition 4.7: An IFS A is called a simple fuzzy region in a connected IFTS, such that:

1) $A^{-}, A_{[]}^{-}, and\ A_{\Diamond}^{-}$ are regular closed,
2) $A^{o}, A_{[]}^{o}, and\ A_{\Diamond}^{o}$ are regular open, and
3) $\partial A, \partial A_{[]}, and\ \partial A_{\Diamond}$ are connected.

Figure 5 illustrates a schematic view of an IF simple region.

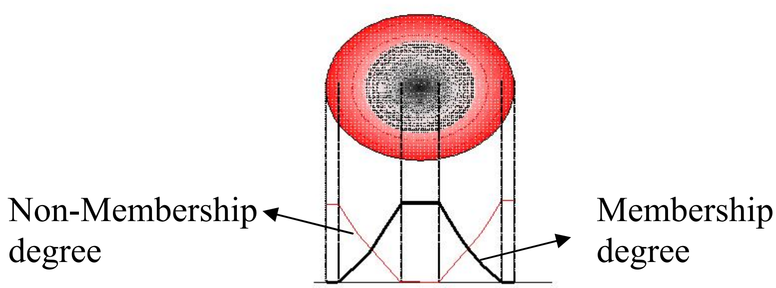


Fig. 5. The representation of a simple IF region.

Having $A^{o}, A_{[]}^{o}, A_{\Diamond}^{o}, A^{E}, \partial A, \partial A_{[]}$, and $\partial A_{\Diamond}$ for two regions, we enable to find spatial relationships between two IFRS.

5 Conclusion and future work

In contrary to the traditional fuzzy logic, IF logic is well equipped to deal with missing data. By employing IFSs in spatial data models, we can express a hesitation concerning the object of interest. This article has gone a step forward in developing methods that can be used to define fuzzy spatial regions and their relationships.

The main contributions of the paper can be described as the following: Possible applications have been listed after the definition of IF. Links to other models have been shown. We are defining some new operators to describe fuzzy objects, describing a simple fuzzy region. This paper has demonstrated that fuzzy spatial object may profitably be addressed in terms of intuitionistic fuzzy logic.

Implementation of the named applications is necessary as a proof of concept. Especially the routing application for hikers is of big interest for us within the GEORAMA project. The finding of spatial relationships and defining complex regions as well as other spatial objects are interesting research topic for the future.

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